CS 170 Homework 15 (Optional)

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, explicitly write "none".

2 Hadamard Gate

The Hadamard Gate is a quantum gate that acts on a single qubit and is represented by the following matrix:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(a) Suppose we pass a 0 qubit through a Hadamard gate, and observe the output on the other end. With what probability will the observed output be 0? With what probability will the observed output be 1?

(b) What if we had passed in a 1 qubit instead?

(c) Briefly state how we might think to simulate this input/output behavior without the use of quantum gates.

(d) Verify that the Hadamard gate acts as its own inverse. What happens if we pass a 0 or 1 qubit through two Hadamard gates, and then observe the output?

(e) Why might the result of part (d) be surprising given the answer to part (c)?

3 (Optional) Entanglement

$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is one of the famous "Bell states," a highly entangled state of its two qubits. In this question, we examine some of its strange properties.

(a) Suppose this Bell state could be decomposed as the (tensor) product of two qubits, the first state in $\alpha_0|0\rangle + \alpha_1|1\rangle$ and the second state in $\beta_0|0\rangle + \beta_1|1\rangle$. Write four equations that the amplitudes α_0, α_1, β_0, and β_1 must satisfy. Conclude that the Bell state cannot be decomposed.

(b) What is the result of measuring the first qubit of $|\psi\rangle$?

(c) What is the result of measuring the second qubit after measuring the first qubit?

(d) If the two qubits in state $|\psi\rangle$ are (physically) very far from each other, can you see why the answer to (c) is surprising?

4 Multiway Cut

In the multiway cut problem, we are given a graph $G = (V, E)$ with k special vertices $s_1, s_2, \ldots, s_k$. Our goal is to find the smallest set of edges F which, when removed from the graph, disconnects the graph into at least k components, where each s_i is in a different component. When $k = 2$, this is exactly the min s-t cut problem, but if $k \geq 3$, the problem becomes NP-hard.

Consider the following algorithm: Let F_i be the set of edges in the minimum cut with s_i on one side and all other special vertices on the other side. Output F, the union of all F_i. Note that this is a multiway cut because removing F_i from G isolates s_i in its own component.

(a) Explain how each F_i can be found in polynomial time.

(b) Let F^* be the smallest multiway cut. Consider the connected components that removing F^* disconnects G into, and let C_i be the set of vertices in the component with s_i. Let F_i^* be the set of edges in F^* with exactly one endpoint in C_i. How many different F_i^* does each edge in F^* appear in? Which is larger: F_i or F_i^*?

(c) Using your answer to the previous part, show that $|F| \leq 2|F^*|$.

(d) How could you modify this algorithm to output F such that $|F| \leq (2 - \frac{2}{k})|F^*|$?

5 Hitting Set

In the Hitting Set Problem, we are given a family of finite integer sets $\{S_1, S_2, \ldots, S_n\}$ and a budget b, and we wish to find an integer set H of size $\leq b$ which intersects every S_i, if such an H exists. In other words, we want $H \cap S_i \neq \emptyset$ for all i.

Show that the Hitting Set Problem is NP-complete. (Hint: Hitting Set generalizes one of the problems covered in Chapter 8 of the textbook).

6 The Matching Game

The matching game is played over the complete weighted bipartite graph $G(V, E)$ with positive edge weights w_e. The edge player plays an edge $e \in E$ while the vertex player plays a vertex $v \in V$, and if v is one of the endpoints of e (we will denote this by $v \in e$), the edge player pays w_e to the vertex player. The edge player would like to minimize the amount they have to pay the vertex player, while the vertex player wants to maximize their earnings.

(a) If the vertex player plays a uniformly random vertex, what is the best response for the edge player?

(b) If the edge player plays a uniformly random edge from the minimum weight matching, what is the best response for the vertex player?

(c) Are these two strategies optimal for this game? If so, provide a brief justification. If not, write the 2 LPs corresponding to the edge player and vertex player, respectively, such that solving them yields each player's optimal strategy. You do not need to write these LPs in canonical form; however, it should be clear what all the constraints are.

7 Motel Choosing

You are traveling along a long road, and you start at location $r_0 = 0$. Along this road, there are n motels at location $\{r_i\}_{i=1}^n$ with $0 < r_1 < r_2 < \cdots < r_n$. The only places you may stop at are these motels, but you can choose which to stop at. You must stop at the final motel (at distance r_n), which is your destination.

Ideally, you want to travel exactly T miles a day and stop at a motel at the end of the day, but this may not be possible (depending on the spacing of the motels). Instead, you receive a *penalty* of $(T - x)^8$ each day, if you travel x miles during the day. The goal is to plan your stops to minimize the total penalty (over all travel days).

Describe and analyze an algorithm that outputs the minimum penalty, given the locations $\{r_i\}$ of the motels and the value of T.

8 The Greatest Roads in America

Arguably, one of the best things to do in America is to take a great American road trip. And in America, there are some amazing roads to drive on (think Pacific Coast Highway, Route 66, etc). An intrepid traveler has chosen to set course across America in search of some amazing driving. What is the length of the shortest path that hits at least k of these amazing roads?

Assume that the roads in America can be expressed as a directed weighted graph $G = (V, E, d)$, and that our traveler wishes to drive across at least k roads from the subset $R \subseteq E$ of "amazing" roads. Furthermore, assume that the traveler starts and ends at her home $a \in V$. You may also assume that the traveler is fine with repeating roads from R, i.e., the k roads chosen from R need not be unique.

Design an efficient algorithm to solve this problem. Provide a 3-part solution with runtime in terms of $n = |V|$, $m = |E|$, and k.

Hint 1: Create a new graph G' based on G such that for some s', t' in G', each path from s' to t' in G' corresponds to a path of the same length from a to itself in G containing at least k roads in R. It may be easier to start by trying to solve the problem for $k = 1$.

9 Three-Legged Race

You are a teacher for two classes, each of n students. You are organizing a three-legged race, where students are paired with students in the opposing class. Each student must be paired up, and each student will only be paired with one other student. In an ideal three-legged race, you and your partner are evenly matched in terms of stride. Luckily, you have data on the stride lengths of all your students. Design an algorithm to minimize the total difference in stride length between pairs.

Please provide an algorithm description, a proof of correctness, and a runtime analysis.

10 Rigged Tournament

Peter is in charge of organizing a football tournament with n teams. The tournament is a single-elimination tournament: if teams i and j play, the team that loses is out of the tournament and cannot play any more games. There are no ties.

Peter's shady friend Jeff has given him the following inside information: if teams i and j play, then they will score a combined total of $f(i, j) \geq 0$ points in that game. Furthermore, Jeff can rig the match so that the team of his choice wins. Peter wishes to find a tournament schedule which (1) maximizes the number of points scored in the tournament, and (2) makes his favorite team, team i^*, win the tournament. Give an efficient algorithm to solve this problem and provide its runtime; proof of correctness is not required.

Note: teams need not play an equal number of games in the tournament. For example, if the teams are $\{1, 2, 3, 4\}$, then a valid tournament schedule (where (i, j) means i plays j and i wins) is $[(1, 2), (1, 3), (4, 1)]$. Here, team 4 wins the tournament.

11 Minimum ∞-Norm Cut

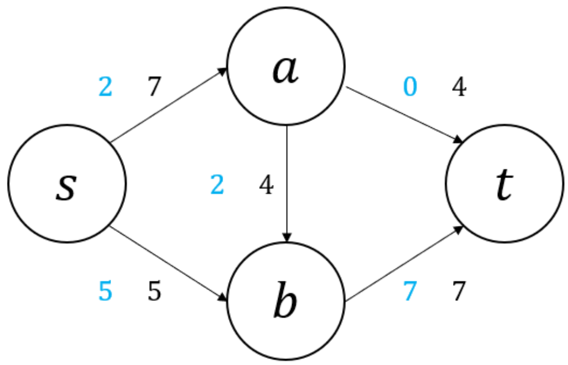
In the MINIMUM INFINITY-NORM CUT problem, you are given a connected undirected graph $G = (V, E)$ with positive edge weights w_e, and you are asked to find a cut in the graph where the largest edge in the cut is as small as possible (note that there is no notion of source or target; any cut with at least one node on each side is valid).

Solve this problem in $O(|E| \log |V| + |V| + |E|)$ time. Please provide an algorithm description, a proof of correctness, and a runtime analysis.

Hint: Minimum Spanning Tree does not require edge weights to be positive.

12 Practice With Residual Graphs

(a) Consider the following network and flow on this network. An edge is labeled with its flow value (in blue) and capacity (in black). e.g., for the edge (s, a), we are currently pushing 2 units of flow on it, and it has a capacity of 7.



Draw the residual graph for this flow.

(b) We are given a network $G = (V, E)$ whose edges have integer capacities c_e, and a maximum flow f from node s to node t. Explicitly, f is given to us in the representation of integer flows along every edge e, $\{f_e\}_{e \in E}$.

However, we find out that one of the capacity values of G was wrong: for edge (u, v), we used c_{uv} whereas it really should have been $c_{uv} - 1$. This is unfortunate because the flow f uses that particular edge at full capacity: $f_{uv} = c_{uv}$. To fix this, we could run Ford-Fulkerson from scratch, but there's a faster way.

Describe an algorithm to fix the max-flow for this network in $O(|V| + |E|)$ time. **Give a three-part solution.**

13 Meal Replacement

Jonny is planning an "Introduction to CS Theory" overnight summer camp for penguins in Antarctica. Penguins can't solve problems well when they're hungry, so Jonny wants to secure an emergency source of food in case polar bears sneak in and eat everything in the igloo. Unfortunately, he is on a tight budget, and in order to accommodate as many penguins as possible, he needs to minimize the cost of food while still meeting the penguins' minimum dietary needs.

Every penguin needs to consume at least 600 calories of protein per day, 800 calories of carbs per day, and 500 calories of fat per day. Jonny has three options for food he's considering buying: salmon, bread, and squid. The composition of each food is provided in the following table:

Food Type	**Price per Pound**	**Protein Calories per Pound**	**Carb Calories per Pound**	**Fat Calories per Pound**
Salmon	6	400	0	150
Bread	1	50	300	25
Squid	8	300	100	200

Our goal is to find a combination of these options that meets the penguins' daily dietary needs while being as cheap as possible.

(a) Formulate this problem as a linear program.

(b) Take the dual of your LP from part (a).

(c) Suppose now there is a pharmacist trying to assign a price to three pills, with the hopes of getting us to buy these pills instead of food. Each pill provides exactly one of protein, carbs, and fat.

Interpret the dual LP variables, objective, and constraints as an optimization problem from the pharmacist's perspective.